

CONTAINMENT PROBLEMS FOR PROJECTIONS OF POLYHEDRA AND SPECTRAHEDRA

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ABSTRACT. Spectrahedra are affine sections of the cone of positive semidefinite matrices which form a rich class of convex bodies that properly contains that of polyhedra. While the class of polyhedra is closed under linear projections, the class of spectrahedra is not. In this paper we investigate the problem of deciding containment of projections of polyhedra and spectrahedra based on previous works on containment of spectrahedra. The main concern is to study these containment problems by formulating them as polynomial nonnegativity problems. This allows to state hierarchies of (sufficient) semidefinite conditions by applying (and proving) sophisticated Positivstellensätze. We also extend results on a solitary sufficient condition for containment of spectrahedra coming from the polyhedral situation as well as connections to the theory of (completely) positive linear maps.

1. INTRODUCTION

A containment problem is the task to decide the set-theoretic inclusion of two given sets. In a broader sense this includes, e.g., radii [13] or packing problems [2]. For some classes of convex sets there has been strong interest in containment problems. This includes containment problems of polyhedra and balls [10] and containment of polyhedra [15]. In recent years, containment problems for spectrahedra, which naturally generalize the class of polyhedra, have seen great interest. Ben-Tal and Nemirovski started that investigation by developing approximations of uncertain linear matrix inequalities yielding a quantitative semidefinite criterion for the so-called *matrix cube problem*, the decision problem whether a cube is contained in a spectrahedron [3]. Helton, Klep, and McCullough studied the geometry of so-called *free spectrahedra* (also known as matricial relaxation of spectrahedra) including containment problems [17, 18]. They established connections to operator theory, namely equivalence between containment of free spectrahedra and positivity of a certain linear map. From that they gained a sufficient semidefinite criterion for containment of spectrahedra which coincides with the Ben-Tal-Nemirovski criterion when applied to the matrix cube problem. Recently Helton, Klep, McCullough, and Schweighofer showed that the quantitative version of Ben-Tal-Nemirovski's criterion is the best possible [19].

The sufficient semidefinite criterion has been reproved by Theobald, Trabant, and the author by a geometric approach. They also provided exact semidefinite characterizations for containment in several important cases [25]. In a second work the authors formulated the containment problem for spectrahedra as a polynomial nonnegativity question [26]. Based on this formulation they studied a hierarchy of semidefinite programs each serving as a sufficient condition for containment coming from Lasserre's moment approach [21] and

Putinar's Positivstellensatz [33]. It turned out that the first step of the hierarchy is implied by the solitary criterion coming from positive linear maps and the geometric approach, yielding finite convergence statements in several cases. In [24] the author considered a different but related hierarchy of semidefinite programs based on the Positivstellensatz by Hol and Scherer [23]. As this approach relies on the geometry of the spectrahedra and the defining linear matrix pencils it allows stronger results. Specifically, all finite convergence results from [26] can be brought forward to this hierarchy and its first step coincides with the solitary criterion. In addition, using the connection to the theory of positive linear maps, finite convergence is shown for a special family of 2-dimensional bounded spectrahedra.

This paper is concerned with containment problems for projections of polyhedra and spectrahedra. Besides the natural question of extending the results for containment of polyhedra and spectrahedra to their projections, the paper is motivated by the growing attention the geometry of projections of polyhedra and spectrahedra attracted in recent years. Among others they have become relevant in many areas like polynomial optimization [5, 12, 31], (real) convex algebraic geometry [6, 20], and extended formulations of polytopes [9].

Starting point of our considerations are the methods and results discussed above. More precisely, we treat possible extensions of the geometric approach, positive linear maps and polynomial optimization to the case of projections. The main considerations and contributions are the following.

- (1) Although the class of polyhedra is closed under (linear) projections, containment problems become more subtle. This is reflected in the fact that the containment problem of two projected polyhedra is co-NP-complete (Theorem 2.4). We formulate the problem as a bilinear nonnegativity question (Theorem 3.1) and study its geometry.
- (2) As the class of spectrahedra is not closed under projections, containment problems are even more subtle. However under additional assumptions (which are common in semidefinite programming) a similar formulation as in the polyhedral case is possible (Theorem 4.1).

Retreating to the case where only one set is given as a projection, allows to bring forward several results from the non-projected case.

- (3) Based on the polyhedral case we deduce a sufficient semidefinite criterion for containment of a projected spectrahedron in a spectrahedron (Theorem 5.5).
- (4) We establish a refinement of Hol-Scherer's Positivstellensatz based on the geometry of the projected sets (Theorem 5.7). That allows to state a hierarchy of sufficient semidefinite conditions to decide containment. The first step of the refined hierarchy coincides with the solitary criterion. As a corollary we gain a Positivstellensatz for polynomials on projections of polytopes (Proposition 5.11).
- (5) The connection between containment and the concept of positive linear maps can be extended to this case (Theorem 5.14).

The paper is structured as follows. After introducing some relevant notation, we state some basics on projections of polyhedra and spectrahedra; see Section 2. We formulate

the containment problem for projections of polyhedra as a polynomial nonnegativity question in Section 3 and extend it to the case of projections of spectrahedra in Section 4. In Section 5 we retreat to the containment problem of a projected spectrahedron in a spectrahedron.

2. PRELIMINARIES

Let \mathcal{S}^k be the space of symmetric $k \times k$ matrices with real entries and \mathcal{S}_+^k be the closed, convex cone of positive semidefinite $k \times k$ matrices. For $x = (x_1, \dots, x_d)$ denote by $\mathcal{S}^k[x]$ the space of symmetric $k \times k$ matrices with entries in the polynomial ring $\mathbb{R}[x]$. For real symmetric matrices $A_0, A_1, \dots, A_d \in \mathcal{S}^k$ a linear matrix polynomial $A(x) = A_0 + \sum_{p=1}^d x_p A_p \in \mathcal{S}^k[x]$ is called a *linear (matrix) pencil*. The positivity domain of $A(x)$ is defined as the set of points in \mathbb{R}^d for which $A(x)$ is positive semidefinite,

$$S_A = \{x \in \mathbb{R}^d \mid A(x) \succeq 0\},$$

where $A(x) \succeq 0$ denotes positive semidefiniteness. The closed, convex, and basic closed semialgebraic set S_A is called a *spectrahedron*.

Every \mathcal{H} -polyhedron $P_A = \{x \in \mathbb{R}^d \mid a + Ax \geq 0\}$ has a natural representation as a spectrahedron called the *normal form* of the polyhedron P_A as a spectrahedron,

$$(2.1) \quad P_A = \left\{ x \in \mathbb{R}^d \mid A(x) = \bigoplus_{i=1}^k a_i(x) = \begin{bmatrix} a_1(x) & & 0 \\ & \ddots & \\ 0 & & a_k(x) \end{bmatrix} \succeq 0 \right\},$$

where $a_i(x) = (a + Ax)_i$ for $i \in [k]$. However the converse is not true, i.e., there exist nondiagonal pencils describing polyhedra. Deciding whether a given spectrahedron is a polyhedron, the so-called *Polyhedrality Recognition Problem* (PRP), is NP-hard [35] and can be reduced to an \mathcal{H} -in- \mathcal{S} containment problem [4]. Note that the normal form used here does not coincide with the normal form used in [25, 26] as we do not require the constant term a to be the all-ones vector.

Following the common notation for bounded polyhedra (*polytopes*), we call a bounded spectrahedron a *spectratope*.

Denote by $\pi : \mathbb{R}^{d+m} \rightarrow \mathbb{R}^d (x, y) \mapsto x$ the linear coordinate projection map. By Fourier-Motzkin elimination, given an \mathcal{H} -polyhedron $P = \{(x, y) \in \mathbb{R}^{d+m} \mid a + Ax + A'y \geq 0\}$, the projection of P onto the x coordinates is again an \mathcal{H} -polyhedron. Unfortunately, a quantifier-free \mathcal{H} -description of $\pi(P)$ can be exponential in the input size (d, m, k) , where k is the number of rows in A and A' ; see [38, Sections 1.2 and 1.3] and the references therein.

Given a linear pencil $A(x, y) \in \mathcal{S}^k[x, y]$ with $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_m)$ for some nonnegative integer m , a projection of the spectrahedron S_A is its image under an affine map. By an elementary observation, without loss of generality, we can assume that the affine projection is a coordinate projection.

Proposition 2.1 ([11, Section 2]). *If a set $T \subseteq \mathbb{R}^d$ is the image of a spectrahedron S under an affine map, then there exists a linear pencil $A(x, y) \in \mathcal{S}^k[x, y]$ with $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_m)$ for some nonnegative integer m such that T is a coordinate projection of*

$S_A \subseteq \mathbb{R}^{d+m}$. Furthermore, if T and S have nonempty interior, then this can be assumed for S_A too.

Due to the proposition, we always assume that the *projection of spectrahedron* S_A as given by the linear pencil $A(x, y) \in \mathcal{S}^k[x, y]$ with $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_m)$, $m \geq 0$, is the set

$$(2.2) \quad \pi(S_A) = \{x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^m : A(x, y) \succeq 0\},$$

where $\pi : \mathbb{R}^{d+m} \rightarrow \mathbb{R}^d$ denotes the coordinate projection.

While projections of polyhedra are again polyhedral, this is not true for spectrahedra (see, e.g., [6, Section 6.3.1]). Moreover, whereas spectrahedra are basic closed semialgebraic sets (the semialgebraic constraints are given by the nonnegativity condition on the principal minors), projected spectrahedra are generally not. Though they are semialgebraic, they are not (basic) closed in general; see Example 4.4.

We state an easy observation for completeness.

Lemma 2.2. *Let $A(x, y) \in \mathcal{S}^k[x, y]$ be a linear pencil.*

- (1) $S_A \neq \emptyset \iff \pi(S_A) \neq \emptyset$.
- (2) *If S_A is bounded, then $\pi(S_A)$ is bounded.*

The converse of part (2) in the previous lemma is not true in general.

Throughout the paper we use the following notation. The class of projections of \mathcal{H} -polyhedra (resp. spectrahedra) is denoted by $\pi\mathcal{H}$ (resp. $\pi\mathcal{S}$). For integers $m, n \in \mathbb{Z}$ with $m \leq n$ we write $[m, n] = \{m, m+1, \dots, n\}$ and $[m] = [1, \dots, m]$.

2.1. Complexity of Containment Problems. The computational complexity of containment problems concerning polyhedra is well-known [10, 14, 15, 16]. Recently this has been extended to spectrahedra [3, 25]. We shortly classify the complexity of several containment problems for projections of \mathcal{H} -polytopes and spectrahedra. For more details on the complexity classification see [24].

Our model of computation is the binary Turing machine: projections of polytopes are presented by certain rational numbers, and the size of the input is defined as the length of the binary encoding of the input data (see, e.g., [13]). Consider the linear projection map $\pi : \mathbb{R}^{d+m} \rightarrow \mathbb{R}^d$, $(x, y) \mapsto x$. An $\pi\mathcal{H}$ -polytope $\pi(P)$ is given by a tuple $(d; m; k; A; A'; a)$ with $d, m, k \in \mathbb{N}$, matrices $A \in \mathbb{Q}^{k \times d}$ and $A' \in \mathbb{Q}^{k \times m}$, and $a \in \mathbb{Q}^k$ such that $\pi(P) = \{x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^m : a + Ax + A'y \geq 0\}$ is bounded. For algorithmic questions, a linear pencil is given by a tuple $(d; m; k; A_0, \dots, A_d, A'_1, \dots, A'_m)$ with $d, m, k \in \mathbb{N}$ and $A_0, \dots, A_d, A'_1, \dots, A'_m \in \mathbb{Q}^{k \times k}$ rational symmetric matrices such that the projected spectrahedron is given by $\pi(S) = \{x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^m : A(x, y) \succeq 0\}$.

Containment questions for spectrahedra are connected to feasibility questions of semidefinite programs in a natural way. A *Semidefinite Feasibility Problem* (SDFP) is defined as the following decision problem (see, e.g., [27, 34]).

$$(2.3) \quad \begin{array}{l} \text{Given } d, k \in \mathbb{N} \text{ and rational symmetric } k \times k\text{-matrices } A_0, A_1, \dots, A_d, \\ \text{decide whether there exists } x \in \mathbb{R}^d \text{ such that } A(x) \succeq 0. \end{array}$$

Equivalently, one can ask whether the spectrahedron S_A is nonempty. Although checking positive semidefiniteness can be done in polynomial time by computing a Cholesky factorization, the complexity classification of the problem SDFP is one of the major open complexity questions related to semidefinite programming (see [8, 34]). Using semidefinite programming techniques, a SDFP can be solved efficiently in practice. In our model of computation, the binary Turing machine, SDFP is known to be feasible in polynomial time if the number of variables d or the matrix size k is fixed [27, Theorem 7].

We first discuss the complexity classification concerning only projections of polytopes.

Theorem 2.3. *Deciding whether a projected \mathcal{H} -polytope is contained in an \mathcal{H} -polytope can be done in polynomial time.*

Proof. Let $\pi(P) = \{x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^m : a + Ax + A'y \geq 0\}$ be a projected \mathcal{H} -polytope and let $Q = \{x \in \mathbb{R}^d \mid b + Bx \geq 0\}$ be an \mathcal{H} -polytope. Embed Q into \mathbb{R}^{d+m} by $Q' = \{(x, y) \in \mathbb{R}^{d+m} \mid b + Bx + 0y \geq 0\}$. Then the containment problem $\pi(P) \subseteq Q$ is equivalent to the \mathcal{H} -in- \mathcal{H} containment problem $P \subseteq Q'$. The statement then follows from [15]. \square

In the latter theorem, the statement does not differ from the non-projected case. The next theorem shows a significant change in the complexity classification when the outer set is a projected \mathcal{H} -polytopes.

Theorem 2.4. *Deciding whether an (projected) \mathcal{H} -polytope is contained in a projected \mathcal{H} -polytope is co-NP-complete.*

Proof. Consider a \mathcal{V} -polytope. It has a representation as the projection of an \mathcal{H} -polytope polynomial in the input data. Thus the containment problem \mathcal{H} -in- $\pi\mathcal{H}$ is co-NP-hard since \mathcal{H} -in- \mathcal{V} is co-NP-complete. It is also in the class co-NP since given a certificate for ' \mathcal{H} not in $\pi\mathcal{H}$ ', i.e., a point p , one can test whether $p \in \mathcal{H}$ and $p \notin \pi\mathcal{H}$ by evaluating the linear constraints of \mathcal{H} (all have to be satisfied) and by solving a *linear feasibility problem* which both is in P by [36, Theorem 13.4]. Therefore \mathcal{H} -in- $\pi\mathcal{H}$ is co-NP-complete. Obviously, the proof remains valid when passing to $\pi\mathcal{H}$ -in- $\pi\mathcal{H}$. \square

In the remaining part, we study the complexity of containment problems involving projections of spectrahedra.

As the complexity of SDFP is unknown, the subsequent statement on containment of a spectrahedron in an \mathcal{H} -polytope does not give a complete answer concerning polynomial solvability of this containment question in the Turing machine model.

Theorem 2.5. *The problem of deciding whether the projection of a spectrahedron is contained in an \mathcal{H} -polytope can be formulated by the complement of semidefinite feasibility problems (involving also strict inequalities), whose sizes are polynomial in the description size of the input data.*

Proof. Consider a spectrahedron S_A given by the linear matrix pencil $A(x, y)$ and the coordinate projection of S_A onto the x -variables $\pi(S_A)$. Given an \mathcal{H} -polytope $P = \{x \in \mathbb{R}^d \mid b + Bx \geq 0\}$ with $b \in \mathbb{Q}^l$ and $B \in \mathbb{Q}^{l \times d}$, construct for each $i \in [l]$ the SDFP

$$(b + Bx)_i < 0, \quad A(x, y) \succeq 0$$

involving a strict inequality. Then $\pi(S_A) \not\subseteq P$ if one of the l SDFPs is not solvable. \square

While the $\pi\mathcal{S}$ -in- \mathcal{H} containment problem is efficiently solvable in practice, the situation changes if the outer set is given as the projection of an \mathcal{H} -polytope.

Theorem 2.6.

- (1) *Deciding whether an (projected) \mathcal{H} -polytope or a (projected) spectrahedron is contained in a (projected) spectrahedron is co-NP-hard.*
- (2) *Deciding whether a (projected) spectrahedron is contained in the projection of an \mathcal{H} -polytope is co-NP-hard.*

Proof. Since the problem \mathcal{H} -in- \mathcal{S} is co-NP-hard (see [3, Proposition 4.1] and [25, Theorem 3.4]), deciding whether a projected \mathcal{H} -polytope or projected spectrahedron is contained in a (projected) spectrahedron is co-NP-hard as well. This is part (1) of the theorem.

Parts (2) is a consequences of Theorem 2.4. \square

2.2. Hol-Scherer's Positivstellensatz. Consider a symmetric matrix polynomial $G = G(x) \in \mathcal{S}^k[x]$ in the variables $x = (x_1, \dots, x_d)$, i.e., a symmetric matrix whose entries lie in the polynomial ring $\mathbb{R}[x]$. We say G has degree t if the maximum degree of the entries is t , i.e., $t = \max\{\deg(G_{ij}) \mid i, j \in [k]\}$.

For matrices $M = (M_{ij})_{i,j=1}^l \in \mathcal{S}^{kl}$ and $N \in \mathcal{S}^k$, define

$$(2.4) \quad \langle M, N \rangle_l := (\langle M_{ij}, N \rangle)_{i,j=1}^l = \sum_{i,j=1}^l E_{ij} \cdot \langle M_{ij}, N \rangle,$$

where E_{ij} denotes the $l \times l$ -matrix with one in the (i, j) th entry and zero otherwise, and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product for matrices. We refer to (2.4) as the *lth scalar product*. It can be seen as a generalization of the Gram matrix representation of a positive semidefinite matrix. Indeed, for positive semidefinite matrices M and N the $l \times l$ -matrix $\langle M, N \rangle_l$ is positive semidefinite as well [22].

For any positive integer l , define the quadratic module generated by $G(x)$

$$(2.5) \quad \mathcal{M}^l(G) = \{S_0(x) + \langle S(x), G(x) \rangle_l \mid S_0(x) \in \Sigma^l[x], S(x) \in \Sigma^{kl}[x]\} \subseteq \mathcal{S}^l[x],$$

where $\Sigma^k[x] \subseteq \mathcal{S}^k$ is the set of sum of squares $k \times k$ -matrix polynomials. A matrix polynomial $S = S(x) \in \mathcal{S}^k[x]$ is called *sum of squares* (sos-matrix for short) if it has a decomposition $S = U(x)U(x)^T$ with $U(x) \in \mathbb{R}^{k \times m}[x]$ for some positive integer m . Equivalently, S has the form $(I_k \otimes [x]_t)^T Z (I_k \otimes [x]_t)$, where $[x]_t$ denotes the monomial basis in x up to $t = \max\{\deg(S_{ij}(x))/2 \mid i, j \in [k]\}$ and Z is a positive semidefinite matrix of appropriate size. For $k = 1$, S is called a sos-polynomial. Checking whether a matrix polynomial is a sos-matrix is an SDFP (2.3). Obviously, every element in $\mathcal{M}^l(G)$ is positive semidefinite on the semialgebraic set $S_G := \{x \in \mathbb{R}^d \mid G(x) \succeq 0\}$. Hol and Scherer [23] showed that for matrix polynomials positive definite on S_G the converse is true under the Archimedeaness condition.

We state the desired Positivstellensatz of Hol and Scherer. See [28] for an alternative proof by Klep and Schweighofer using the concept of pure states.

Proposition 2.7 ([23, Corollary 1]). *Let l be a positive integer and let $S_G = \{x \in \mathbb{R}^d \mid G(x) \succeq 0\}$ for a matrix polynomial $G \in \mathcal{S}^k[x]$. If the quadratic module $\mathcal{M}^l(G)$ is Archimedean, then it contains every matrix polynomial $F \in \mathcal{S}^l[x]$ positive definite on S_G .*

By restricting to diagonal matrix polynomials G and $l = 1$, one gets the Positivstellensatz of Putinar [33] as a corollary.

Corollary 2.8. *Let $G = \{g_1, \dots, g_k\} \subseteq \mathbb{R}[x]$ and $S_G = \{x \in \mathbb{R}^d \mid g \geq 0 \ \forall g \in G\}$. If the quadratic module*

$$M(G) = \left\{ s_0(x) + \sum_{i=1}^k s_i(x)g_i(x) \mid s_0, s_1, \dots, s_m \in \Sigma^1[x] \right\}$$

is Archimedean, then it contains every polynomial $f \in \mathbb{R}[x]$ positive on S_G .

Interestingly, the usual quadratic module $\mathcal{M}^1(G)$ is Archimedean if and only if $\mathcal{M}^l(G)$ is for any positive integer l .

Proposition 2.9. *The following two statements are equivalent.*

- (1) *For some positive integer l , the quadratic module $\mathcal{M}^l(G)$ is Archimedean.*
- (2) *For all positive integers l , the quadratic module $\mathcal{M}^l(G)$ is Archimedean.*

Furthermore, assume G is a linear pencil. Then $\mathcal{M}^l(G)$ for any positive integer l is Archimedean if and only if the spectrahedron S_G is bounded.

The equivalence of the first two statements was proved by Helton, Klep, and McCullough for monic linear matrix pencils in the language of their matricial relaxation; see [18, Lemma 6.9]. We recapitulate the proof and extend it to quadratic modules generated by arbitrary matrix polynomials.

Proof. The implication (2) \implies (1) is obvious. To show the reverse implication, note first that $\mathcal{M}^l(G)$ is Archimedean if and only if $(N - x^T x)I_l \in \mathcal{M}^l(G)$ for some positive integer N . Let $m \in \mathbb{N}$ be arbitrary but fixed. We have to show that $(N - x^T x)I_m \in \mathcal{M}^m(G)$. Denote by E_{11} the $m \times m$ -matrix with one in the entry $(1, 1)$ and zero elsewhere and let Q be the $l \times m$ -matrix with one in the entry $(1, 1)$ and zero elsewhere. Clearly, $E_{11} = Q^T Q$. Let $(N - x^T x)I_l = S_0 + \langle S, G \rangle_l$ with $S = (S_{ij})_{i,j=1}^l$ be the desired sos-representation. Setting $\tilde{S}_0 := Q^T S_0 Q = (S_0)_{11} E_{11} \in \Sigma^m[x]$ and $\tilde{S} = E_{11} \otimes S_{11} \in \Sigma^m[x]$, we get

$$(N - x^T x)E_{11} = Q^T (N - x^T x)I_l Q = Q^T (S_0 + \langle S, G \rangle_l) Q = \tilde{S}_0 + E_{11} \langle S_{11}, G \rangle = \tilde{S}_0 + \langle \tilde{S}, G \rangle_m.$$

Applying the same to E_{ii} for $i \in [m]$ and using additivity of the quadratic module $\mathcal{M}^m(G)$ yields $(N - x^T x)I_m \in \mathcal{M}^m(G)$.

The last statement follows from [30, Corollary 4.4.2] (see also [29]) together with the shown equivalence. \square

3. A BILINEAR FORMULATION OF THE $\pi\mathcal{H}$ -IN- $\pi\mathcal{H}$ CONTAINMENT PROBLEM

For $a \in \mathbb{R}^k$, $A \in \mathbb{R}^{k \times d}$, $A' \in \mathbb{R}^{k \times m}$ and $b \in \mathbb{R}^l$, $B \in \mathbb{R}^{l \times d}$, $B' \in \mathbb{R}^{l \times n}$ let

$$(3.1) \quad \begin{aligned} \pi(P_A) &= \{x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^m : a + Ax + A'y \geq 0\} \\ \text{and } \pi(P_B) &= \{x \in \mathbb{R}^d \mid \exists y' \in \mathbb{R}^n : b + Bx + B'y' \geq 0\} \end{aligned}$$

be projections of the \mathcal{H} -polyhedra P_A and P_B , respectively. Note that both $\pi(P_A)$ and $\pi(P_B)$ are \mathcal{H} -polyhedra themselves (and thus closed sets). A quantifier-free \mathcal{H} -description however can be exponential in the input size (d, m, k) respectively (d, n, l) ; cf. Section 2.

Our starting point is the formulation of the containment problem as a bilinear feasibility problem. Interestingly, the projection variables y' of the outer polyhedron do not appear in the feasibility system (or the optimization version below) only the corresponding coefficients B' .

Theorem 3.1. *Let $\pi(P_A)$ and $\pi(P_B)$ be as defined in (3.1) and $\pi(P_A)$ be nonempty.*

(1) *$\pi(P_A)$ is contained in $\pi(P_B)$ if and only if*

$$z^T(b + Bx) \geq 0 \quad \text{on} \quad \pi(P_A) \times (\ker(B'^T) \cap \mathbb{R}_+^l).$$

(2) *Let $\ker(B'^T) \cap \mathbb{R}_+^l = \text{span}(B')^\perp \cap \mathbb{R}_+^l \neq \{0\}$. Then $\pi(P_A) \subseteq \pi(P_B)$ if and only if*

$$z^T(b + Bx) \geq 0 \quad \text{on} \quad \pi(P_A) \times (\ker(B'^T) \cap \Delta^l),$$

where $\Delta^l = \{z \in \mathbb{R}^l \mid \mathbf{1}_l^T z = 1, z \geq 0\}$ is the l -simplex.

The additional assumption on the kernel of B'^T seems to be somewhat artificial, however, if the projection of P_B to the x -coordinates is bounded, then the condition holds. The two main advantages of part (2) in Theorem 3.1 are the boundedness of the z variables and that the condition $z^T(b + Bx) \geq 0$ is indeed an inequality. (Note that in part (1), containment is equivalent to $z^T(b + Bx) \equiv 0$ on $\pi(P_A) \times (\ker(B'^T) \cap \mathbb{R}_+^l)$ as $(x, z) = (x, 0)$ is a feasible solution for all $x \in \pi(P_A)$.) The next lemma serves as a first step in a geometric interpretation of this precondition.

Lemma 3.2. *Let $\pi(P_B)$ be as in (3.1). Then $\ker(B'^T) \cap \mathbb{R}_+^l = \text{span}(B')^\perp \cap \mathbb{R}_+^l = \{0\}$ if and only if $\text{span}(B') \cap \mathbb{R}_{++}^l \neq \emptyset$. In this case, $\pi(P_B) = \mathbb{R}^d$.*

In particular, if $\pi(P_B)$ is bounded, then $\ker(B'^T) \cap \mathbb{R}_+^l = \text{span}(B')^\perp \cap \mathbb{R}_+^l \neq \{0\}$.

Proof. The equivalence $\ker(B'^T) \cap \mathbb{R}_+^l = \text{span}(B')^\perp \cap \mathbb{R}_+^l = \{0\} \iff \text{span}(B') \cap \mathbb{R}_{++}^l \neq \emptyset$ is easy to see. If so, then there exists $y' \in \mathbb{R}^n$ such that $B'y' > 0$. Thus, for every $x \in \mathbb{R}^d$, there exists $t > 0$ sufficiently large such that $b + Bx + B'(ty') \geq 0$. This implies $\pi(P_B) = \mathbb{R}^d$. Thus for bounded $\pi(P_B)$ we have $\ker(B'^T) \cap \mathbb{R}_+^l \neq \{0\}$. \square

Before proving Theorem 3.1, we observe that neither the implication “ $\text{span}(B') \cap \mathbb{R}_{++}^l \neq \emptyset \implies \pi(P_B) = \mathbb{R}^d$ ” nor the implication “ $\pi(P_B)$ is bounded $\implies \ker(B'^T) \cap \mathbb{R}_+^l \neq \{0\}$ ” in Lemma 3.2 is an equivalence. Example 3.3 also shows that the precondition in part (2) of Theorem 3.1 cannot be dropped.

Example 3.3.

(1) Consider the polyhedron

$$P_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} y \geq 0 \right\}.$$

P_1 is a pointed polyhedral cone containing the origin in its interior; see Figure 1 (A). We have $\text{span}(B') \cap \mathbb{R}_{++}^2 \neq \emptyset$ and thus the intersection of $\ker(B'^T) = \ker(1, 1)$ and the

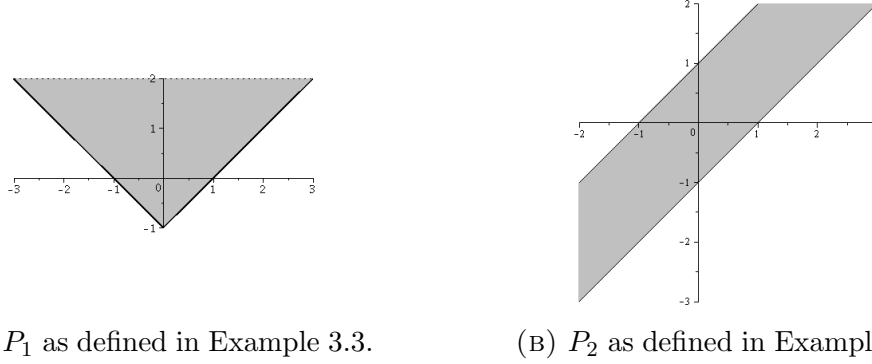


FIGURE 1

nonnegative real numbers is zero-dimensional, i.e., $\ker(B'^T) \cap \mathbb{R}_+^2 = \{0\}$. Moreover, in this case, the restriction to the 1-simplex as in part (2) of Theorem 3.1 is not possible.

(2) Consider the polyhedron

$$P_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} y \geq 0 \right\},$$

which is unbounded and contains the origin in its interior; see Figure 1 (B). We have $\text{span}(B') \cap \mathbb{R}_{++}^2 = \emptyset$ and thus $\ker(B'^T) \cap \mathbb{R}_+^2 \neq \{0\}$. Indeed, for every $t \geq 0$, we have $(0, t) \in \ker(B'^T) \cap \mathbb{R}_+^2$. On the other hand, $\pi(P_B) = \mathbb{R}$ shows that the reverse of the other (and above mentioned) implications in Lemma 3.2 are not equivalences.

Proof of Theorem 3.1.

(1): $\pi(P_A) \not\subseteq \pi(P_B)$ if and only if there exists a point $x \in \pi(P_A) \setminus \pi(P_B)$, i.e., for $x \in \pi(P_A)$ there exists no $y' \in \mathbb{R}^n$ with $b + Bx + B'y' \geq 0$. By Farkas' Lemma [38, Proposition 1.7] this is equivalent to the existence of a point $z \in \mathbb{R}_+^l$ with $z^T B' = 0$ such that $z^T(b + Bx) < 0$ holds. Equivalently, there exists $(x, z) \in \pi(P_A) \times (\ker(B'^T) \cap \mathbb{R}_+^l)$ such that $z^T(b + Bx) < 0$.

(2): If there exists $(x, z) \in \pi(P_A) \times (\ker(B'^T) \cap \mathbb{R}_+^l)$ such that $z^T(b + Bx) < 0$, then $z \neq 0$ and thus $z'^T(b + Bx) < 0$ for $z' = \frac{z}{|z|} \geq 0$ with $|z'| = \sum_{i=1}^l z'_i = \frac{1}{|z|} \sum_{i=1}^l z'_i = 1$.

Assume $z^T(b + Bx) \geq 0$ holds for all $(x, z) \in \pi(P_A) \times (\ker(B'^T) \cap \mathbb{R}_+^l)$. By assumption, there exists $0 \neq z \in \ker(B'^T) \cap \mathbb{R}_+^l$. Applying the same scaling as above yields $z^T(b + Bx) \geq 0$ for every $z \in \Delta^l$, implying the claim. \square

Note that $\ker(B'^T) \cap \Delta_+^l$ is a polytope and is intrinsically linked to the polar of $\pi(P_B)$. Namely, it is the set of convex combinations of the columns in B'^T that are equal to the origin, i.e., $0 = B'^T z$ with $1 = \mathbf{1}_l^T z$ and $z \geq 0$.

Consider the optimization version of Theorem 3.1

$$(3.2) \quad \begin{aligned} & \inf \quad z^T(b + Bx) \\ & \text{s.t.} \quad (x, y, z) \in P_A \times (\ker(B'^T) \cap \Delta_+^l). \end{aligned}$$

Assuming nonemptiness of $\ker(B'^T) \cap \Delta_+^l$, Theorem 3.1 implies that $\pi(P_A) \subseteq \pi(P_B)$ if and only if the infimum is nonnegative.

Replacing the nonnegativity constraints in (3.2) by sos constraints results in a hierarchy of SDFPs to decide the $\pi\mathcal{H}$ -in- $\pi\mathcal{H}$ containment problem,

$$(3.3) \quad \begin{aligned} \mu(t) &:= \sup \mu \\ \text{s.t. } & z^T(b + Bx) - \mu \in M^1 + I, \end{aligned}$$

where M^1 and I denote the quadratic module generated by the inequality constraints and the ideal generated by the equality constraints, respectively.

Under the assumptions in part (2) of Theorem 3.1, applying Putinar's Positivstellensatz, Corollary 2.8, to problem (3.3) (also allowing equality constraints), the sequence $\mu(t)$ converges asymptotically to the optimal value of (3.2) for $t \rightarrow \infty$.

4. A BILINEAR FORMULATION OF THE $\pi\mathcal{S}$ -IN- $\pi\mathcal{S}$ CONTAINMENT PROBLEM

Throughout the section, let

$$\begin{aligned} A(x, y) &= A_0 + \sum_{i=1}^d A_i x_i + \sum_{j=1}^m A'_j y_j \in \mathcal{S}^k[x, y] \\ \text{and } B(x, y') &= B_0 + \sum_{i=1}^d B_i x_i + \sum_{j=1}^n B'_j y'_j \in \mathcal{S}^l[x, y'] \end{aligned}$$

be linear pencils with $y = (y_1, \dots, y_m)$ and $y' = (y'_1, \dots, y'_n)$ for $n \geq 1$. Denote the projection of the corresponding spectrahedra onto the x -variables by

$$\begin{aligned} \pi(S_A) &= \{x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^m : A(x, y) \succeq 0\} \\ \text{and } \pi(S_B) &= \{x \in \mathbb{R}^d \mid \exists y' \in \mathbb{R}^n : B(x, y') \succeq 0\}. \end{aligned}$$

Recall from Section 2 that the projection of a spectrahedron is not necessarily closed and thus, in general, not a spectrahedron itself.

Define $\bar{\mathcal{B}} = \text{span}\{B'_1, \dots, B'_n\}$ and recall the equivalence

$$\langle B'_i, Z \rangle = 0 \ \forall i \in [n] \iff Z \in \bar{\mathcal{B}}^\perp.$$

The $\pi\mathcal{S}$ -in- $\pi\mathcal{S}$ containment problem is slightly more involved than the $\pi\mathcal{H}$ -in- $\pi\mathcal{H}$ problem as the projection of a spectrahedron fails to be closed in general. We state an extension of Theorem 3.1 to the $\pi\mathcal{S}$ -in- $\pi\mathcal{S}$ containment problem.

Theorem 4.1. *Let $A(x, y) \in \mathcal{S}^k[x, y]$ and $B(x, y') \in \mathcal{S}^l[x, y']$ be linear pencils such that $\pi(S_A) \neq \emptyset$.*

- (1) $\pi(S_A) \subseteq \text{cl } \pi(S_B)$ if and only if $\langle B(x, 0), Z \rangle \geq 0$ on $\pi(S_A) \times (\bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l)$.
- (2) Assume that the condition $\sum_{i=1}^n B'_i y'_i \succeq 0 \implies \sum_{i=1}^n B'_i y'_i = 0$ holds for all y' . Then the closure in part (1) can be dropped.

As in the $\pi\mathcal{H}$ -in- $\pi\mathcal{H}$ problem, the projection variables y' of the outer spectrahedron do not appear in the polynomial formulation, only the corresponding coefficient matrices.

We use the following Farkas type lemmas to prove Theorem 4.1.

Lemma 4.2 ([37, Theorem 2.22]). *Let $A(x) \in \mathcal{S}^k[x]$ be a linear pencil and denote by $\tilde{A}(x) = \sum_{i=1}^d x_i A_i$ the pure-linear part. Then exactly one of the following two systems has a solution.*

$$(4.1) \quad \forall \varepsilon > 0 \exists A'_0 \in \mathcal{S}^k, \exists x \in \mathbb{R}^d : \|A_0 - A'_0\| < \varepsilon, A'_0 + \tilde{A}(x) \in \mathcal{S}_+^k$$

$$(4.2) \quad \exists Z \in \mathcal{S}^k : Z \succeq 0, \langle A_i, Z \rangle = 0 \forall i \in [d], \langle A_0, Z \rangle < 0$$

The Farkas type lemmas for cones (and thus the theory of semidefinite programming) lack in the fact that the linear image of the cone of positive semidefinite matrices is not closed in general. Additional conditions which lead to more clean formulations are called *constraint qualification*.

Lemma 4.3 ([7, Example 5.14]). *Let $A(x) \in \mathcal{S}^k[x]$ be a linear pencil. Assume*

$$\sum_{i=1}^d A_i x_i \succeq 0 \implies \sum_{i=1}^d A_i x_i = 0$$

holds for any x . Then either (4.2) has a solution or S_A is nonempty.

If A_1, \dots, A_d are linearly independent, then the above condition can be replaced by $\sum_{i=1}^d A_i x_i \succeq 0 \implies x = 0$.

Proof of Theorem 4.1.

(1): Assume $\pi(S_A) \subseteq \text{cl } \pi(S_B)$. Let $x \in \pi(S_A)$. Then there exists a sequence $(x_i, y'_i)_i \subseteq S_B$ such that $\lim_{i \rightarrow \infty} x_i = x$. For all $Z \in \bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l$ it holds that

$$\langle B(x, 0), Z \rangle = \lim_{i \rightarrow \infty} \langle B(x_i, y'_i), Z \rangle \geq 0.$$

Since $x \in \pi(S_A)$ is arbitrary, $\langle B(x, 0), Z \rangle$ is nonnegative on $\pi(S_A) \times (\bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l)$.

Assume $\langle B(x, 0), Z \rangle \geq 0$ on $\pi(S_A) \times (\bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l)$. Let $x \in \pi(S_A)$ be fixed but arbitrary and set $B'_0 = B(x, 0)$. By Lemma 4.2, there exist $B''_0 \in \mathcal{S}^l$ and $y' \in \mathbb{R}^n$ such that $B''_0 + \sum_{i=1}^n B'_i y'_i \in \mathcal{S}_+^l$ and $\|B'_0 - B''_0\| < \varepsilon$ for all $\varepsilon > 0$. By letting ε tend to zero, there exists a sequence $(y'_\varepsilon)_\varepsilon \subseteq \mathbb{R}^n$ such that $\lim_{\varepsilon \rightarrow 0} B(x, y'_\varepsilon) \succeq 0$. As $x \in \pi(S_A)$ is arbitrary, the claim follows.

(2): Assume $\langle B(x, 0), Z \rangle \geq 0$ on $\pi(S_A) \times (\bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l)$. Let $x \in \pi(S_A)$ be fixed but arbitrary. By Lemma 4.3, the spectrahedron $\{y' \in \mathbb{R}^n \mid B'_0 + \sum_{i=1}^n B'_i y'_i \succeq 0\}$ is nonempty. Thus there exists $y' \in \mathbb{R}^n$ such that $B(x, y') \succeq 0$. \square

Unfortunately, the if-part in Theorem 4.1 (1) without taking the closure is generally not true as the next example shows.

Example 4.4. Consider the linear pencil

$$B(x, y') = \begin{bmatrix} -y'_1 & x & 0 \\ x & 1 - y'_2 & 0 \\ 0 & 0 & -x + y'_2 \end{bmatrix} = \begin{bmatrix} 0 & x & 0 \\ x & 1 & 0 \\ 0 & 0 & -x \end{bmatrix} + y'_1 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + y'_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and let $A(x)$ be the univariate linear pencil

$$A(x) = \begin{bmatrix} 1 - x & 0 \\ 0 & 1 + x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

describing the interval $S_A = [-1, 1]$. By inspecting the principal minors of B , the spectrahedron S_B has the form $\{(x, y') \in \mathbb{R}^3 \mid y'_1 \leq 0, x \leq y'_2 \leq 1, y'_1(1 - y'_2) + x^2 \leq 0\}$. For $x = 1$, the second condition implies $y'_2 = 1$ and thus the third condition reads as $x^2 \leq 0$, a contradiction. Thus $S_A \not\subseteq \pi(S_B) = (-\infty, 1)$.

For every $Z \in \bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^3$ it holds that

$$0 = \langle Z, B'_1 \rangle = -Z_{11} \implies Z_{12} = 0, \quad 0 = \langle Z, B'_2 \rangle = Z_{33} - Z_{22}$$

implying $\langle B(x, 0), Z \rangle = Z_{22} + x(-Z_{33} + 2Z_{12}) = Z_{22}(1 - x) \geq 0$ for all $x \in S_A$.

It should not be surprising that the constraint qualification on the pencil $B(x, y')$ is not satisfied. Indeed, for $(y'_1, y'_2) = (y'_1, 0)$ with $y'_1 < 0$,

$$B'_1 y'_1 + B'_2 y'_2 = \begin{bmatrix} -y'_1 & x & 0 \\ x & -y'_2 & 0 \\ 0 & 0 & -x + y'_2 \end{bmatrix} = \begin{bmatrix} -y'_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is positive semidefinite but not identically zero.

An issue when considering the practical utility of Theorem 4.1 is the unboundedness of the set $\bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l$. Under an analog condition as in Theorem 3.1, \mathcal{S}_+^l can be replaced by the spectrahedral analog of the simplex.

Corollary 4.5. *Let $A(x, y) \in \mathcal{S}^k[x, y]$ and $B(x, y') \in \mathcal{S}^l[x, y']$ be linear pencils such that $\pi(S_A) \neq \emptyset$. Assume $\bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l \neq \{0\}$. Then $\pi(S_A) \subseteq \text{cl } \pi(S_B)$ if and only if*

$$\langle B(x, 0), Z \rangle \geq 0 \text{ on } \pi(S_A) \times (\bar{\mathcal{B}}^\perp \cap \mathbb{T}^l),$$

where $\mathbb{T}^l = \{Z \in \mathcal{S}_+^l \mid \langle I_l, Z \rangle = 1\}$ is the l -spectraplex.

Proof. Since $\bar{\mathcal{B}}^\perp \cap \mathbb{T}^l \subseteq \bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l$, the “only if”-part follows from Theorem 4.1.

For the converse, first suppose there exists $(x, Z) \in \pi(S_A) \times (\bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l)$ such that $\langle B(x, 0), Z \rangle < 0$. Then $0 \neq Z \in \mathcal{S}_+^l$ and thus $\text{tr}(Z) = \langle I_l, Z \rangle > 0$. This implies $\langle B(x, 0), Z' \rangle < 0$ for $Z' = \frac{Z}{\text{tr}(Z)}$ with $\text{tr}(Z') = \langle I_l, Z' \rangle = \frac{1}{\text{tr}(Z)} \langle I_l, Z \rangle = 1$.

Assume $\langle B(x, 0), Z \rangle \geq 0$ on $\pi(S_A) \times (\bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l)$. By assumption, there exists $0 \neq Z \in \bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l = \bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l$. Applying the above scaling, the claim follows. \square

We state an analogue to Lemma 3.2. As the proof is very similar, we skip it here.

Lemma 4.6. *We have $\bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l = \{0\}$ if and only if $\mathcal{B} \cap \mathcal{S}_{++}^l \neq \emptyset$. In this case, $\pi(S_B) = \mathbb{R}^d$. In particular, if $\pi(S_B)$ is bounded, then $\bar{\mathcal{B}}^\perp \cap \mathcal{S}_+^l \neq \{0\}$.*

Restricting the $\pi\mathcal{S}$ -in- $\pi\mathcal{S}$ containment problem to the special case $\pi\mathcal{S}$ -in- $\pi\mathcal{H}$ allows to state improved versions of Theorem 4.1 and Corollary 4.5.

Proposition 4.7. *Let $\pi(P_B)$ be as in (3.1) and let $A(x, y) \in \mathcal{S}^l[x, y]$ be a linear pencil.*

- (1) $\pi(S_A) \subseteq \pi(P_B)$ if and only if $z^T(b + Bx) \geq 0$ on $\pi(S_A) \times (\ker(B'^T) \cap \mathbb{R}_+^l)$.
- (2) Assume $\ker(B'^T) \cap \mathbb{R}_+^l \neq \{0\}$. Then $\pi(S_A) \subseteq \pi(P_B)$ if and only if $z^T(b + Bx) \geq 0$ on $\pi(S_A) \times (\ker(B'^T) \cap \Delta^l)$.

Proof. $\pi(S_A) \not\subseteq \pi(P_B)$ if and only if there exists $x \in \pi(S_A)$ such that $\nexists y' \in \mathbb{R}^n : b + Bp + B'y' \geq 0$. By Farkas' Lemma [38, Proposition 1.7] this is equivalent to the existence of a $z \in \mathbb{R}_+^l$ with $z^T B' = 0$ and $z^T(b + Bp) < 0$. The claims follow as in the proofs of Theorem 4.1 and Corollary 4.5. \square

We close with an example.

Example 4.8. Let M be the convex hull of the shifted unit disks defined by the identities $1 - (x_1 + 1)^2 - x_2^2 = 0$ and $1 - (x_1 - 1)^2 - x_2^2 = 0$, respectively. M is the projection of a spectrahedron. Indeed, considering only the first disk and shifting it along the segment $[-1, 1] \times \{0\}$ yields $M = \{x \in \mathbb{R}^2 \mid \exists y \in \mathbb{R} : 1 - (x - y)^2 - x_2^2 \geq 0, -1 \leq y \leq 1\}$. It is the projection of the 3-dimensional cylinder, see Figure 2, defined by the linear pencil

$$A(x, y) = \begin{bmatrix} 1 - x_2 & x_1 - y \\ x_1 - y & 1 + x_2 \end{bmatrix} \oplus \begin{bmatrix} 1 - y & 0 \\ 0 & 1 + y \end{bmatrix}.$$

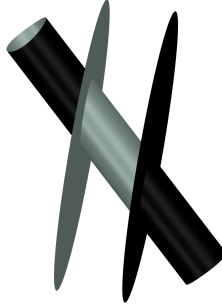


FIGURE 2. The determinantal variety of $A(x, y)$ with S_A being the grey cylinder in the middle of the picture; see Example 4.8.

The so-called TV screen (see, e.g., [6, Section 6.3.1]) is the projection of the spectrahedron

$$S_B = \left\{ (x, y) \in \mathbb{R}^{2+2} \mid A(x, y) = \begin{bmatrix} 1 + y_1 & y_2 \\ y_2 & 1 - y_1 \end{bmatrix} \oplus \begin{bmatrix} 1 & x_1 \\ x_1 & y_1 \end{bmatrix} \oplus \begin{bmatrix} 1 & x_2 \\ x_2 & y_2 \end{bmatrix} \succeq 0 \right\},$$

onto the x variables; see Figure 3.

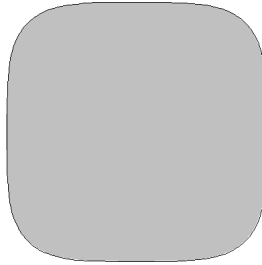


FIGURE 3. The TV screen as stated in Example 4.8.

Both $\pi(S_A)$ and $\pi(S_B)$ are closed but not spectrahedra. We have

$$\bar{\mathbb{B}}^\perp \cap \mathbb{T}^6 = \left\{ Z \in \mathcal{S}_+^6 \mid \sum_{i=1}^6 Z_{ii} = 1, Z_{22} = Z_{11} + Z_{44}, Z_{66} = -2Z_{12} \right\}.$$

For all $Z \in \bar{\mathbb{B}}^\perp \cap \mathbb{T}^6$ the objective $\langle B(x, 0), Z \rangle$ in Corollary 4.5 can then be written as

$$\langle B(x, 0), Z \rangle = 1 - Z_{44} - Z_{66} + 2x_1 Z_{34} + 2x_2 Z_{56}.$$

We want to find a pair $(x^*, Z^*) \in \pi(S_A) \times (\bar{\mathbb{B}}^\perp \cap \mathbb{T}^6)$ such that $\langle B(x, 0), Z \rangle < 0$. To this end, consider $x^* = (1 + \varepsilon, 0) \in \pi(S_A)$ for all $\varepsilon \in [0, 1]$ and

$$Z^* = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \oplus \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \oplus 0_{3 \times 3} \in \mathcal{S}_+^6.$$

Then $\langle B(x^*, 0), Z^* \rangle = -\frac{2}{3}\varepsilon < 0$ for all $\varepsilon \in (0, 1]$. Thus $\pi(S_A) \not\subseteq \pi(S_B)$.

Now interchange the roles of S_A and S_B , i.e., $\pi(S_A)$ is the TV screen and $\pi(S_B)$ is the convex hull of two disks. Then $\bar{\mathbb{B}}^\perp \cap \mathbb{T}^4$ is the set

$$\bar{\mathbb{B}}^\perp \cap \mathbb{T}^4 = \left\{ Z \in \mathcal{S}_+^4 \mid \sum_{i=1}^4 Z_{ii} = 1, 2Z_{12} = Z_{44} - Z_{33} \right\}.$$

For all $Z \in \bar{\mathbb{B}}^\perp \cap \mathbb{T}^4$, $\langle B(x, 0), Z \rangle$ has the form

$$\langle B(x, 0), Z \rangle = (1 - x_2)Z_{11} + (1 + x_2)Z_{22} + (1 - x_1)Z_{33} + (1 + x_1)Z_{44}.$$

As $1 \pm x_i \geq 0$ for all $x \in \pi(S_A)$, we have $\langle B(x, 0), Z \rangle \geq 0$ for all $(x, Z) \in \pi(S_A) \times (\bar{\mathbb{B}}^\perp \cap \mathbb{T}^4)$. Thus $\pi(S_A) \subseteq \pi(S_B)$.

5. SUM OF SQUARES CERTIFICATES FOR THE $\pi\mathcal{S}$ -IN- \mathcal{S} CONTAINMENT PROBLEM

Retreating to the cases $\pi\mathcal{H}$ -in- \mathcal{H} and $\pi\mathcal{S}$ -in- \mathcal{S} allows to bring forward several results from the non-projected case. We start with the polyhedral situation in Theorem 5.1. It also serves as an algorithmic proof of Theorem 2.3. Afterwards, we state and prove a sophisticated Positivstellensatz for the second problem.

5.1. From the $\pi\mathcal{H}$ -in- \mathcal{H} to the $\pi\mathcal{S}$ -in- \mathcal{S} Containment Problem. As the proofs of the statements in this section are similar to the ones given in [25], we only stress the emerging differences in the proofs.

Even in the non-projected case, i.e., $m = 0$, Theorem 5.1 below is a slight extension of a statement in [25]. Namely, here we drop the conditions $a = \mathbf{1}_k$, and $b = \mathbf{1}_l$ as well as the boundedness condition.

Theorem 5.1. *Consider the polyhedra $P_A = \{(x, y) \in \mathbb{R}^{d+m} \mid a + Ax + A'y \geq 0\} \neq \emptyset$ and $P_B = \{x \in \mathbb{R}^d \mid b + Bx \geq 0\}$.*

- (1) $\pi(P_A) \subseteq P_B$ if and only if there exists a nonnegative matrix $C \in \mathbb{R}_+^{l \times k}$ and a nonnegative vector $c_0 \in \mathbb{R}_+^l$ with $b = c_0 + Ca$, $B = CA$, and $0 = CA'$.
- (2) Let P_A be a polytope that is not a singleton. Then $\pi(P_A) \subseteq P_B$ if and only if there exists a nonnegative matrix $C \in \mathbb{R}_+^{l \times k}$ with $b = Ca$, $B = CA$, and $0 = CA'$.

Testing whether P_A is a singleton is easy as one has to check that the system of equalities $a + Ax = 0$ has a single solution. Certainly, in this situation, checking containment is trivial as $\pi(P_A) \subseteq P_B$ is equivalent to test whether a single point has nonnegative entries. The precondition in part (2) of Theorem 5.1, however, cannot be removed in general; see part (1) of Example 5.2.

For unbounded polyhedra the additional term c_0 is required in order for the criterion to be exact. Without it, already in the simple case of two half spaces defined by two parallel hyperplanes, the restriction of the condition in part (1) of Theorem 5.1 to part (2) can fail to be feasible; see part (2) of Example 5.2.

Example 5.2.

(1) Consider the polytopes P_A and P_B given by the systems of linear inequalities

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x \geq 0 \text{ and } \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ -1 & 1 \end{bmatrix} x \geq 0,$$

respectively. P_A is the singleton $\{(1, 0)\}$ and P_B is a simplex containing P_A . There is no matrix C satisfying the conditions in part (2) of Theorem 5.1 (with $m = 0$). Indeed, $b = Ca$ implies $0 = C_{11} - C_{12}$ and $B = CA$ implies $1 = B_{11} = (-C_{11} + C_{12}, -C_{11})$, a contradiction. A solution to the linear feasibility system in Theorem 5.1 (1) is given by

$$c_0 = \mathbf{1}_3, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Moreover, it is easy to see that for any $P_B = \{x \in \mathbb{R}^2 \mid b + Bx \geq 0\}$ containing P_A containment is certified if and only if B has the form $B = [-b, -b + c]$ for some vector c .

(2) Consider the half space given by the linear polynomial $a(x) = 1 - x_1 - x_2$. Let $b(x) = b + [B_1, B_2]x$ be any half space. The condition in part (2) of the Theorem 5.1 is satisfied if and only if $b = c$, $B_1 = -c$, $B_2 = -c$ for $c \geq 0$. Thus either $b(x) \equiv 0$ or $b(x)$ is a positive multiple of $a(x)$.

To prove Theorem 5.1 we use the following affine form of Farkas' Lemma.

Lemma 5.3 ([36, Corollary 7.1h]). *Let $P = \{x \in \mathbb{R}^d \mid a + Ax \geq 0\}$ be nonempty. Then every affine polynomial $f \in \mathbb{R}[x]$ nonnegative on P can be written as $f(x) = c_0 + \sum_{i=1}^m c_i(a + Ax)_i$ with nonnegative coefficients c_i .*

Proof of Theorem 5.1. If $B = CA$, $0 = CA'$, and $b = Ca$ (resp. $b = c_0 + Ca$) with a nonnegative matrix C , for any $x \in \pi(P_A)$ we have

$$b + Bx + 0y = C(a + Ax + A'y) \geq 0,$$

i.e., $\pi(P_A) \subseteq P_B$.

Conversely, if $\pi(P_A) \subseteq P_B$, then any of the linear polynomials $(b + Bx + 0y)_i$, $i \in [l]$, is nonnegative on P_A . Hence, by Lemma 5.3, $(b + Bx + 0y)_i$ can be written as a linear

combination

$$(b + Bx + 0y)_i = c'_{i0} + \sum_{j=1}^k c'_{ij}(a + Ax + A'y)_j$$

with nonnegative coefficients c'_{ij} . Comparing coefficients yields $b_i = c'_{i0} + \sum_{j=1}^k c'_{ij}$ for $i \in [l]$, implying part (1) of the statement.

To prove the second part, first translate both P_A and P_B to the origin. By assumption, there exists $(\bar{x}, \bar{y}) \in P_A$. Define $\bar{a} := a + A\bar{x} + A'\bar{y}$ and $\bar{b} := b + B\bar{x}$. Then $\bar{a} \geq 0$ and $0 \in \{x \in \mathbb{R}^d \mid \bar{a} + Ax + A'y \geq 0\}$, implying

$$\bar{b} = C\bar{a}, \quad B = CA, \quad 0 = CA' \iff b = Ca, \quad B = CA, \quad 0 = CA'.$$

Thus w.l.o.g. let $a \geq 0$.

Stiemke's Transposition Theorem [36, Section 7.8] implies the existence of a $\lambda > 0$ such that $[A^T, A'^T]\lambda = 0$, and thus

$$\lambda^T(a + Ax + A'y) = \lambda^T a = 1$$

after an appropriate rescaling. Note that $a \neq 0$ as otherwise $P_A = \{0\}$ is a singleton. By multiplying that equation with c'_{i0} from above, we obtain nonnegative c''_{ij} with $\sum_{j=1}^k c''_{ij}(a + Ax + A'y)_j = c'_{i0}$, yielding

$$(b + Bx)_i = \sum_{j=1}^k (c'_{ij} + c''_{ij})(a + Ax + A'y)_j.$$

Hence, $C = (c_{ij})_{i,j=1}^k$ with $c_{ij} := c'_{ij} + c''_{ij}$ is a nonnegative matrix with $B = CA$, $0 = CA'$, and $(Ca)_i = \sum_{j=1}^k (c'_{ij} + c''_{ij})a_j = b_i - c'_{i0} + c'_{i0} \lambda^T a = b_i$ for every $i \in [l]$. \square

The sufficiency part of Theorem 5.1 can be extended to the case of projected spectrahedra via the normal form (2.1) of a (projected) polyhedron P_A as a (projected) spectrahedron,

$$\pi(P_A) = \{x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^m : A(x, y) = \text{diag}(a_1(x, y), \dots, a_k(x, y)) \succeq 0\},$$

where $a_i(x, y)$ is the i th entry of the vector $a + Ax + A'y$.

Corollary 5.4. *Let $A(x, y) \in \mathcal{S}^k[x, y]$ and $B(x) \in \mathcal{S}^l[x]$ be normal forms of polyhedra (2.1).*

(1) $\pi(S_A) \subseteq S_B$ if and only if there exist positive semidefinite diagonal matrices C_0, C such that

$$(5.1) \quad B_0 = C_0 + \sum_{i=1}^k (A_0)_{ii} C_{ii}, \quad B_p = \sum_{i=1}^k (A_p)_{ii} C_{ii} \quad \forall p \in [d], \quad 0 = \sum_{i=1}^k (A'_p)_{ii} C_{ii} \quad \forall p \in [m].$$

(2) Let S_A be a polytope that is not a singleton. $\pi(S_A) \subseteq S_B$ if and only if system (5.1) has a solution with $C_0 = 0$.

If the diagonality condition on the matrix C in Corollary 5.4 is dropped, then the above SDFP yields a sufficient condition for the $\pi\mathcal{S}$ -in- \mathcal{S} containment problem. Subsequently, the indeterminate matrix $C = (C_{ij})_{i,j=1}^k$ is a symmetric $kl \times kl$ -matrix, where the C_{ij} are $l \times l$ -blocks.

Theorem 5.5. *Let $A(x, y) \in \mathcal{S}^k[x, y]$ and $B(x) \in \mathcal{S}^l[x]$ be linear pencils. Denote by $\pi(S_A)$ the coordinate projection of the spectrahedron S_A . If there exist positive semidefinite matrices $C = (C_{ij})_{i,j=1}^k \in \mathcal{S}_+^{kl}$ and $C_0 \in \mathcal{S}_+^l$ such that*

$$(5.2) \quad B_0 = C_0 + \sum_{i,j=1}^k (A_0)_{ij} C_{ij}, \quad B_p = \sum_{i,j=1}^k (A_p)_{ij} C_{ij} \quad \forall p \in [d], \quad 0 = \sum_{i,j=1}^k (A'_p)_{ij} C_{ij} \quad \forall p \in [m],$$

then $\pi(S_A) \subseteq S_B$.

In the non-projected case, the sufficient semidefinite criterion (5.2) has first been developed by Helton et al. [18] using the theory of positive linear maps (cf. Section 5.4) and has been reproved in [25] by elementary methods. In [24] the author showed that the condition is exactly the 0th step of the hierarchy based on truncation of the Hol-Scherer quadratic module (2.5). Here we are bringing this forward to the projected case.

For completeness we state a short proof of Theorem 5.5 based on [25].

Proof. We have

$$B(x) = B_0 + \sum_{p=1}^d x_p B_p = C_0 + \sum_{i,j=1}^k (A(x, y))_{ij} C_{ij} = C_0 + \mathbb{I}^T ((A(x, y))_{ij} C_{ij})_{i,j=1}^k \mathbb{I}$$

with $\mathbb{I} = [I_1, \dots, I_l]^T \in \mathbb{R}^{kl \times l}$. Let $x \in \pi(S_A)$. By definition, there exists $y \in \mathbb{R}^m$ such that $A(x, y) \succeq 0$. Thus the Kronecker product $A(x, y) \otimes C$ is positive semidefinite. Since $((A(x, y))_{ij} C_{ij})_{i,j=1}^k$ is a principal submatrix of $A(x, y) \otimes C$, we have $B(x) \succeq 0$ as well. \square

Even for the non-projected case, the sufficient semidefinite criterion (5.2) is not necessary for containment in general; see [25, Section 6.1].

5.2. A Sophisticated Positivstellensatz. Consider the linear pencils $A(x, y) \in \mathcal{S}^k[x]$ and $B(x) \in \mathcal{S}^l[x]$. Then $\pi(S_A)$ is contained in S_B if and only if $B(x) \succeq 0$ on $\pi(S_A)$. If S_A is a spectratope, then this is equivalent to $B(x) + \varepsilon I_l \in \mathcal{M}^l(A)$ for all $\varepsilon > 0$, where

$$\mathcal{M}^l(A) = \{S_0 + \langle S, A(x, y) \rangle_l \mid S_0 \in \Sigma^l[x, y], \quad S \in \Sigma^{kl}[x, y]\}$$

is the quadratic module associated to $A(x, y)$ as defined in (2.5). Clearly, if $B(x) \in \mathcal{M}^l(A)$ for a linear pencil $B(x) \in \mathcal{S}^l[x]$, then $\text{cl } \pi(S_A) \subseteq S_B$. Thus truncation of the $\mathcal{M}^l(A)$ yields a hierarchy of SDFPs to decide $\pi\mathcal{S}$ -in- \mathcal{S} containment.

The drawback of this approach to the $\pi\mathcal{S}$ -in- \mathcal{S} containment problem is that it relies on the geometry of the spectrahedron S_A rather than its projection, namely the boundedness assumption on S_A and the appearance of the projection variables y in the quadratic module. In the following, we address this by developing a refinement of Hol-Scherer's Positivstellensatz. Particularly, we can eliminate the variables y in the sense that they neither appear in the quadratic module nor in the relaxation.

Gouveia and Netzer [11] derived a Positivstellensatz for polynomials positive on the closure of a projected spectrahedron.

Proposition 5.6 ([11, Theorem 5.1]). *Let $A(x, y) \in \mathcal{S}^k[x, y]$ be a strictly feasible linear pencil. Define the quadratic module*

$$\mathcal{M}(\pi A) = \{s_0 + \langle S, A(x, 0) \rangle \mid \langle S, A'_i \rangle = 0 \ \forall i \in [m], \ s_0 \in \Sigma[x], \ S \in \Sigma^k[x]\}.$$

If $\pi(S_A)$ is bounded, then $\mathcal{M}(\pi A)$ is Archimedean and contains all polynomials positive on the closure of $\pi(S_A)$.

Subsequently, we state and proof an extension to linear pencils positive definite on a projected spectrahedron. Thereto define the quadratic module

$$(5.3) \quad \mathcal{M}^l(\pi A) = \{S_0 + \langle S, A(x, 0) \rangle_l \mid \langle S, A'_i \rangle_l = 0 \ \forall i \in [m], \ S_0 \in \Sigma^l[x], \ S \in \Sigma^{kl}[x]\}.$$

It is easy to see that $\mathcal{M}^l(\pi A)$ is in fact a quadratic module. Note that $\mathcal{M}^l(\pi A)$ does not have to be finitely generated; see [11, Section 5]. Clearly, every element of $\mathcal{M}^l(\pi A)$ is positive semidefinite on the closure of $\pi(S_A)$.

Theorem 5.7. *Let $A(x, y) \in \mathcal{S}^k[x, y]$ be a strictly feasible linear pencil such that $\pi(S_A)$ is bounded. For $l \in \mathbb{N}$ the quadratic module $\mathcal{M}^l(\pi A)$ is Archimedean and contains every matrix polynomial positive definite on $\text{cl } \pi(S_A)$.*

Proof. By boundedness of $\pi(S_A)$ there exists $N \in \mathbb{N}$ sufficiently large such that $N \pm x_i$ is nonnegative on $\pi(S_A)$ for all $i \in [d]$. We show that under the preconditions in the theorem $\mathcal{M}^l(\pi A)$ contains every linear polynomial nonnegative on $\pi(S_A)$. Then the quadratic module is Archimedean.

Let $b(x) = b_0 + b^T x \in \mathbb{R}[x]_1$ be a fixed but arbitrary affine linear polynomial nonnegative on $\pi(S_A)$. Consider the following primal-dual pair of SDPs.

$$\begin{aligned} p^* := \inf \quad & b(x) \\ \text{s.t.} \quad & A(x, y) \succeq 0 \end{aligned} \qquad \begin{aligned} \sup \quad & \langle -A_0, Z \rangle \\ \text{s.t.} \quad & \langle A_i, Z \rangle = b_i \ \forall i \in [d] \\ & \langle A'_i, Z \rangle = 0 \ \forall i \in [m] \\ & Z \in \mathcal{S}_+^k \end{aligned}$$

Since $A(x, y)$ is strictly feasible by assumption, the dual problem (on the right-hand side) has optimal value $p^* - b_0$ and attains it; see [8, Theorem 2.2]. Since $b(x) \geq 0$ on $\pi(S_A)$, we have $-b_0 \leq p^* - b_0$ and thus

$$b_0 - z_0 = \langle A_0, Z \rangle, \ \langle A_i, Z \rangle = b_i \ \forall i \in [d], \ \langle A'_i, Z \rangle = 0 \ \forall i \in [m]$$

for some $Z \in \mathcal{S}_+^k$ and $z_0 \geq 0$. Define $S(x)$ as the block diagonal $kl \times kl$ -matrix with l copies of Z on its diagonal, i.e., $S(x) = \bigoplus_{j=1}^l Z$, and $S_0(x) = z_0 I_l$. Then

$$S_0(x) + \langle S(x), A(x, 0) \rangle_l = z_0 I_l + \bigoplus_{j=1}^l \langle Z, A(x, 0) \rangle_l = b(x) I_l$$

and $\langle S(x), A'_i \rangle_l = \bigoplus_{j=1}^l \langle Z, A'_i \rangle = 0$ for $i \in [m]$. This implies $b(x) \in \mathcal{M}^l(\pi A)$. By Hol-Scherer's Theorem, every matrix polynomial positive definite on $\text{cl } \pi(S_A)$ is contained in $\mathcal{M}^l(\pi A)$. \square

Theorem 5.7 leads to a refined hierarchy for the $\pi\mathcal{S}$ -in- \mathcal{S} containment problem using the truncated quadratic module

$$(5.4) \quad \mathcal{M}_t^l(\pi A) = \{S_0 + \langle S, A(x, 0) \rangle_l \mid \langle S, A'_i \rangle_l = 0 \ \forall i \in [m], \ S_0 \in \Sigma_t^l[x], \ S \in \Sigma_t^{kl}[x]\}.$$

It is evident from the definition of the quadratic modules $\mathcal{M}^l(A)$ and $\mathcal{M}^l(\pi A)$ that the latter approach is preferable to the naive way from the theoretical viewpoint (provided that $A(x, y)$ is strictly feasible).

Corollary 5.8. *Let $A(x, y) \in \mathcal{S}^k[x, y]$ be a strictly feasible linear pencil such that $\pi(S_A)$ is bounded and let $B(x) \in \mathcal{S}^l[x]$ be a linear pencil.*

- (1) $\pi(S_A) \subseteq S_B$ if and only if $B(x) + \varepsilon I_l \in \mathcal{M}^l(\pi A)$ for all $\varepsilon > 0$.
- (2) If $B(x) \succ 0$ on $\pi(S_A)$, then $B(x) \in \mathcal{M}^l(\pi A)$.

The 0-th step of the hierarchy based on (5.4) is exactly the sufficient containment criterion stated in Theorem 5.5. Indeed, for $t = 0$, the constant sos-matrix S equals the positive semidefinite matrix C after permuting rows and columns simultaneously.

Proposition 5.9. *Let $A(x, y) \in \mathcal{S}^k[x, y]$ and $B(x) \in \mathcal{S}^l[x]$ be linear pencils. Assume $A(x, y)$ is strictly feasible. The following are equivalent.*

- (1) $B(x) \in \mathcal{M}_0^l(\pi A)$.
- (2) There exist $C' \in \mathcal{S}_+^{kl}$ and $C_0' \in \mathcal{S}_+^l$ such that

$$B_0 = C_0' + \langle A_0, C' \rangle_l, \ B_p = \langle A_p, C' \rangle_l \ \forall p \in [d], \ 0 = \langle A'_q, C' \rangle_l \ \forall q \in [m].$$

- (3) There exist $C \in \mathcal{S}_+^{kl}$, $C_0 \in \mathcal{S}_+^l$ such that

$$B_0 = C_0 + \sum_{i,j=0}^k (A_0)_{ij} C_{ij}, \ B_p = \sum_{i,j=0}^k (A_p)_{ij} C_{ij} \ \forall p \in [d], \ 0 = \sum_{i,j=0}^k (A'_q)_{ij} C_{ij} \ \forall q \in [m].$$

Proof. The equivalence of (1) and (2) follows from the definition of the truncated quadratic module by rewriting it as an SDFP. Applying a simultaneous permutation of the rows and columns of C' in (2) (resp. of C in (3)), the linear systems can easily be transformed.

For details in the non-projected case see [24, Theorem 5.1.11] \square

The proof of Theorem 5.7 evidently yields necessity for the $\pi\mathcal{S}$ -in- \mathcal{H} containment problem. This, in particular, shows the (theoretical) effectiveness of the approach based on Theorem 5.7.

Theorem 5.10. *Let $A(x, y) \in \mathcal{S}^k[x, y]$ be a strictly feasible linear pencil and let the coefficients of the linear pencil $B(x) \in \mathcal{S}^l[x]$ be simultaneously congruent to a diagonal matrix.*

- (1) $\pi(S_A) \subseteq S_B$ if and only if $B(x) \in \mathcal{M}_0^l(\pi A)$.
- (2) Assume S_B is a polytope with nonempty interior. Then $\pi(S_A) \subseteq S_B$ if and only if $B(x) \in \mathcal{M}_0^l(\pi A)$ with $S_0 = 0$.

In particular, the statements (1) and (2) hold for a diagonal linear pencil $B(x)$, i.e., a polyhedron in normal form (2.1).

In order to prove Theorem 5.10, we use natural adaptations of auxiliary results on the behavior of the sufficient containment criterion with regard to block diagonalization and transitivity as shown in [25] to the non-projected setting. Using Proposition 5.9, it is easy to verify the validity of these statements.

Proof. As for $t = 0$ the resulting SDFP is invariant under non-singular congruence transformations of $B(x)$ (see [24, Lemma 5.1.14]), we can retreat to the normal form (2.1) $B(x) = \bigoplus_{q=1}^l b^q(x) \in \mathcal{S}^l[x]$ with $b^q(x) = b_0^q + x^T b^q$ for $q \in [l]$. Denote by $b_0^q, b_1^q, \dots, b_d^q$ the coefficients of the linear form $b^q(x) = (b_0 + Bx)_q$. Set $b^q := (b_1^q, \dots, b_d^q)$.

The proof of Theorem 5.7 yields certificates

$$b_0^q - z_0^q = \langle A_0, Z^q \rangle, \quad \langle A_i, Z^q \rangle = b_i^q \quad \forall i \in [d], \quad \langle A'_i, Z^q \rangle = 0 \quad \forall i \in [m]$$

for some $Z^q \in \mathcal{S}_+^k$ and $z_0^q \geq 0$. Setting $S(x) = \bigoplus_{q=1}^l Z^q$ and $S_0(x) = \bigoplus_{q=1}^l z_0^q$, this implies part (1) of the statement.

To prove the second part, let $S(x)$ as before and set $S_0(x)$ to be zero. Then

$$\langle S(x), A(x, 0) \rangle_l = \bigoplus_{q=1}^l \langle A(x, 0), Z^q \rangle_l = \bigoplus_{q=1}^l \left(f_0 - z_0^q + \sum_{i=1}^d b_i^q x_i \right)$$

certifies the containment $\pi(S_A) \subseteq S_{B'}$, where $B'(x)$ is defined as

$$B'(x) = \bigoplus_{q=1}^l \left(r^q + \sum_{p=1}^d x_p \right).$$

Assuming that S_B is a polytope, we have $S_{B'} \subseteq S_B$ and thus, by transitivity and exactness of the initial hierarchy step for polytopes, see Corollary 5.4, there is a certificate for the containment question $\pi(S_A) \subseteq S_B$ of degree zero with $S_0(x) = 0$. \square

As a special case of Theorem 5.7, we gain a Positivstellensatz for polynomials on projected polyhedra having boundedness as its only precondition.

Proposition 5.11. *Let $P_A = \{(x, y) \in \mathbb{R}^{d+m} \mid a + Ax + A'y \geq 0\}$ be a nonempty polyhedron such that $\pi(P_A)$ is bounded. The quadratic module*

$$\mathcal{M}^1(\pi, A) = \left\{ s_0 + \sum_{i=1}^k s_i(x)(a + Ax)_i \mid \sum_{i=1}^k s_i(x)(A'_{i,j}) = 0 \quad \forall j \in [m], \quad s_0, \dots, s_k \in \Sigma[x] \right\}$$

is Archimedean and contains every polynomial positive on $\pi(P_A)$.

Proof. The proof follows from the proof of Theorem 5.7 by retreating to diagonal pencils and the fact that strong duality holds for linear programming [36, Corollary 7.1g]. \square

5.3. Examples. We discuss some academic examples for the hierarchy stated in (5.4). All computations are made on a desktop computer with Intel Core i3-2100 @ 3.10 GHz and 4 GB of RAM. In the tables, “time” states the time in seconds for setting up the problem in YALMIP [32] and solving it with Mosek [1].

$\pi(S_A)$	rS_B	r	time	$\mu(0)$
two disks	2-ball	1.99	0.7978	-0.0050
		2	0.8215	$5.9978 \cdot 10^{-08}$
		2.01	0.9173	0.0050
S_A	rS_B			
	3-ball	2.23	0.8690	-0.0027
		2.2361	0.7470	$1.4339 \cdot 10^{-05}$
		2.24	0.8803	0.0018

TABLE 1. Computational test of containment as described in Example 5.12.

In the examples we consider the optimization version of (5.4)

$$\begin{aligned} \mu(t) = \sup \quad & \mu \\ \text{s.t.} \quad & B(x) - \mu I_l \in \mathcal{M}_t^l(\pi A). \end{aligned}$$

Letting t tend to infinity, the sequence of optimal values $\mu(t)$ converges to the value $\mu^* = \sup\{\mu \mid B(x) - \mu I_l \succeq 0 \ \forall x \in \pi(S_A)\}$ which is nonnegative if and only if $\pi(S_A) \subseteq S_B$. Thus a nonnegative value $\mu(0)$ states the existence of a containment certificate with $t = 0$.

A d -dimensional *ball* is a spectratope S_A given by the linear pencil

$$(5.5) \quad A(x) = I_{d+1} + \sum_{p=1}^d \frac{x_p}{r} (E_{p,d+1} + E_{d+1,p}) \in \mathcal{S}^{d+1}[x]$$

with $r > 0$.

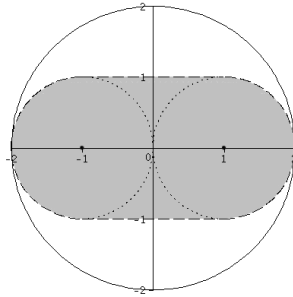


FIGURE 4. The convex hull of two disks in a 2-ball as stated in Example 5.12.

Example 5.12. Consider the convex hull of two disks $\pi(S_A)$ as defined in Example 4.8 and the 2-ball of radius $r > 0$. It follows from the construction of $\pi(S_A)$ that it is centrally symmetric and that its circumradius is 2; see Figure 4. Up to numerical accuracy, this value is computed by our approach; see Table 1.

$\pi(S_A)$	rS_B	r	time	$\mu(0)$
TV screen	2-ball	1.18	0.8514	-0.0078
		$\sqrt[4]{2}$	2.0564	$-1.3621 \cdot 10^{-08}$
		1.19	0.9964	$6.6628 \cdot 10^{-04}$
		1.2	0.9854	0.0090
S_A	rS_B			
	4-ball	1.55	1.1036	-0.0024
		$\sqrt{\sqrt{2} + 1}$	0.9373	$3.1037 \cdot 10^{-09}$
		1.56	1.0723	0.0040

TABLE 2. Computational test of containment as described in Example 5.13.

Example 5.13. Consider the TV screen $\pi(S_A)$ as defined in Example 4.8. Note that while the TV screen is centrally symmetric (as its boundary equals the variety defined by the polynomial $1 - x_1^4 - x_2^4$), its defining spectrahedron is not (as the point $(1, 0, 1, 0)$ is contained in S_A but its negative $(-1, 0, -1, 0)$ is not).

As one can see in Table 2, the circumradius of the TV screen is at most $\sqrt[4]{2} \approx 1.1892$, while the “centrally symmetric circumradius” of S_A is at most $\sqrt{\sqrt{2} + 1} \approx 1.5538$.

Actually, the computed values for the (centrally symmetric) circumradius are exact. For $p = \left(\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \in S_A$ we have $\|\pi(p)\|_2 = \sqrt[4]{2}$ and $\|p\|_2 = \sqrt{\sqrt{2} + 1}$, implying that the circumradius of the TV screen is at least $\sqrt[4]{2}$ and that $\sqrt{\sqrt{2} + 1}$ is the smallest possible radius of a ball (centered at the origin) containing S_A .

5.4. Containment of Projected Spectrahedra and Positive Linear Maps. We discuss an extension of the connection between positive linear maps and containment of spectrahedra (as introduced by Helton et al. [18]; see also [26]) to projected spectrahedra.

Given two linear pencils $A(x, y) \in \mathcal{S}^k[x, y]$ and $B(x) \in \mathcal{S}^l[x]$ with $y = (y_1, \dots, y_m)$ define the linear subspaces

$$\mathcal{A} = \text{span}\{A_0, \dots, A_d, A'_1, \dots, A'_m\} \text{ and } \mathcal{B} = \text{span}\{B_0, \dots, B_d\}.$$

Every element in \mathcal{A} can be associated to a homogeneous linear pencil $A(x_0, x, y) \in \mathcal{S}^k[x_0, x, y]$ (A_0 being the coefficient of x_0). The linear pencil

$$\widehat{A}(x_0, x, y) := x_0(1 \oplus A_0) + \sum_{p=1}^d x_p(0 \oplus A_p) + \sum_{q=1}^m y_q(0 \oplus A'_q)$$

is called the *extended linear pencil* associated to $A(x_0, x, y)$. The associated linear subspace is $\widehat{\mathcal{A}} = \text{span}\{1 \oplus A_0, 0 \oplus A_1, \dots, 0 \oplus A_d, 0 \oplus A'_1, \dots, 0 \oplus A'_m\}$.

For linearly independent $A_1, \dots, A_d, A'_1, \dots, A'_m$, let $\widehat{\Phi}_{AB} : \widehat{\mathcal{A}} \rightarrow \mathcal{B}$ be the linear map defined by

$$\widehat{\Phi}_{AB}(1 \oplus A_0) = B_0, \quad \widehat{\Phi}_{AB}(0 \oplus A_p) = B_p \quad \forall p \in [d], \quad \widehat{\Phi}_{AB}(0 \oplus A'_q) = 0 \quad \forall q \in [m].$$

Since every linear combination $0 = \lambda_0(1 \oplus A_0) + \sum_{p=1}^d \lambda_p(0 \oplus A_p) + \sum_{q=1}^m \lambda_{d+q}(0 \oplus A'_q)$ for real scalars $\lambda_0, \dots, \lambda_{d+m}$ yields $\lambda_0 = 0$, it suffices to assume the linear independence of the coefficient matrices $A_1, \dots, A_d, A'_1, \dots, A'_m$ to ensure that $\widehat{\Phi}_{AB}$ is well-defined. If $A_0, A_1, \dots, A_d, A'_1, \dots, A'_m$ are linearly independent, then we can retreat to the simpler map $\Phi_{AB} : \mathcal{A} \rightarrow \mathcal{B}$ defined by

$$\Phi_{AB}(A_p) = B_p \quad \forall p \in [d] \quad \text{and} \quad \Phi_{AB}(A'_q) = 0 \quad \forall q \in [m].$$

The next theorem extends the key connection between operator theory and containment of spectrahedra to the setting of projections of spectrahedra.

Theorem 5.14. *Let $A(x, y) \in \mathcal{S}^k[x, y]$ and $B(x) \in \mathcal{S}^l[x]$ be linear pencils.*

- (1) *If Φ_{AB} or $\widehat{\Phi}_{AB}$ is positive, then $\pi(S_A) \subseteq S_B$.*
- (2) *If $\pi(S_A) \neq \emptyset$, then $\pi(S_A) \subseteq S_B$ implies positivity of $\widehat{\Phi}$.*
- (3) *If $\pi(S_A) \neq \emptyset$ and S_A is bounded, then $\pi(S_A) \subseteq S_B$ implies positivity of Φ .*

Proof.

(1): Let Φ_{AB} be positive. For every $x \in \pi(S_A)$ there exists $y \in \mathbb{R}^m$ such that $A(x, y) \succeq 0$, i.e., $A(x, y) \in \mathcal{S}_+^k \cap \mathcal{A}$. Then $B(x) = B(x) + \sum_{q=1}^m y_q \Phi(A'_q) = \Phi(A(x, y)) \in \mathcal{S}_+^l \cap \mathcal{B}$ and hence $x \in S_B$. There is no difference in the proof if $\widehat{\Phi}_{AB}$ is positive.

(2): Since the spectrahedra defined by $A(x, y)$ and $\widehat{A}(x, y)$ coincide, their projections equal and hence $\pi(S_{\widehat{A}}) \subseteq S_B$. Let $\widehat{A}(x_0, x, y) \in \mathcal{S}_+^{k+1} \cap \widehat{\mathcal{A}}$. Then $x_0 \geq 0$.

Case $x_0 > 0$. By scaling the linear pencil with $1/x_0$ the positive semidefiniteness is preserved. Thus, $1/x_0 \widehat{A}(x_0, x, y) = \widehat{A}(1, x/x_0, y/x_0) \in \mathcal{S}_+^{k+1} \cap \widehat{\mathcal{A}}$ and $x/x_0 \in \pi(S_A) \subseteq S_B$. Scaling $B(x/x_0)$ by x_0 yields $\widehat{\Phi}(\widehat{A}(x_0, x, y)) = x_0 B_0 + \sum_{p=1}^d x_p B_p = x_0 B(x/x_0) \in \mathcal{S}_+^l \cap \mathcal{B}$.

Case $x_0 = 0$. If $(x, x_0) = (0, 0)$, the statement is obvious. Let $x \neq 0$. Fix a point $\bar{x} \in \pi(S_A) \neq \emptyset$. Then, for some $\bar{y}, y \in \mathbb{R}^m$, $\widehat{A}(1, \bar{x} + tx, \bar{y} + ty) = \widehat{A}(1, \bar{x}, \bar{y}) + \widehat{A}(0, tx, ty) \succeq 0$ for all $t > 0$, implying $\bar{x} + tx \in \pi(S_A) \subseteq S_B$ for all $t > 0$. Thus x lies in the recession cone of $\pi(S_A)$ which clearly is contained in the recession cone of S_B . Indeed, $\frac{1}{t} B(1, \bar{x}) + B(0, x) = \frac{1}{t} B(\bar{x} + tx) \succeq 0$ for all $t > 0$. By closedness of the cone of positive semidefinite matrices, we get $B(0, x) \succeq 0$. Hence, $\widehat{\Phi}(\widehat{A}(x_0, x, y)) = \widehat{\Phi}(\widehat{A}(0, x, y)) = B(0, x) \succeq 0$.

(3): Let $A(x_0, x, y) = x_0 A_0 + \sum_{p=1}^d x_p A_p + \sum_{q=1}^m y_q A'_q \in \mathcal{S}_+^k \cap \mathcal{A}$.

Case $x_0 > 0$. This case follows by a similar scaling argument as in part (2).

Case $x_0 \leq 0$. Since $\pi(S_A) \neq \emptyset$, there exists $\bar{x} \in \pi(S_A)$ and hence, for some $\bar{y} \in \mathbb{R}^m$,

$$A(0, x + |x_0|\bar{x}, y + |x_0|\bar{y}) \succeq |x_0| \cdot A(1, \bar{x}, \bar{y}) \succeq 0.$$

For $A(0, x + |x_0|\bar{x}, y + |x_0|\bar{y}) \neq 0$, one has an improving ray of the spectrahedron S_A , in contradiction to boundedness of S_A . For $A(0, x + |x_0|\bar{x}, y + |x_0|\bar{y}) = 0$, linear independence of A_0, \dots, A_{d+m} implies $(x + |x_0|\bar{x}, y + |x_0|\bar{y}) = (0, 0)$. But then $x_0 A(1, \bar{x}, \bar{y}) = A(x_0, x, y) \succeq$

0 together with $x_0 \leq 0$ and $A(1, \bar{x}, \bar{y}) \succeq 0$ imply either $A(1, \bar{x}, \bar{y}) = 0$, in contradiction to linear independence, or $(x_0, x) = (0, 0)$. Clearly, in this case, $\Phi_{AB}(0) = 0$. \square

The linear map $\widehat{\Phi}_{AB}$ can be represented by an symmetric $(k+1)l \times (k+1)l$ matrix \widehat{C} . By expecting the linear equations defining \widehat{C} , it is easy to see that $\widehat{C} = C_0 \oplus C$, where the matrix pair C_0, C is from Theorem 5.5. Thus Proposition 5.9 implies the next corollary to Theorem 5.14.

Corollary 5.15. *Let $A(x, y) \in \mathcal{S}^k[x, y]$ and $B(x) \in \mathcal{S}^l[x]$ be linear pencils with $A(x, y)$ strictly feasible. Then the following are equivalent.*

- (1) *The map $\widehat{\Phi}_{AB}$ is completely positive, i.e., $\widehat{C} \succeq 0$.*
- (2) *The solitary criterion (5.2) is feasible.*
- (3) *$B(x) \in \mathcal{M}_0^l(\pi A)$.*

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